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Modular Bootstrap of Boundary $\mathcal{N}=2$ Liouville Theory*

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Abstract

We present our recent studies on the dynamics of boundary $\mathcal{N}=2$ Liouville theory. We use the representation theory of $\mathcal{N}=2$ superconformal algebra and the method of modular bootstrap to derive three classes of boundary states of the $\mathcal{N}=2$ Liouville theory. Class 1 and 2 branes are analogues of ZZ and FZZT branes of $\mathcal{N}=0,1$ Liouville theory while class 3 branes come from $U(1)$ degrees of freedom. We compare our results with those of $SL(2;R)/U(1)$ supercoset which is known to be T-dual to $\mathcal{N}=2$ Liouville theory and describes the geometry of 2d black hole. We find good agreements with known results in $SL(2;R)/U(1)$ theory obtained by semi-classical analysis using DBI action. We also comment on the duality of $\mathcal{N}=2$ Liouville theory.

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1 Introduction

Recently there has been a revival of interests in Liouville field theory, due mainly to the reinterpretation of the old results of matrix model-Liouville theory as a typical example of gauge gravity correspondence [1, 2, 3, 4]. Matrix theory is interpreted as describing open string degrees of freedom living on D-branes while the Liouville field describes closed string degrees of freedom. An exact map between the extended D-brane of Liouville theory and macroscopic loop operator of matrix theory, for instance, has been discovered. Analyses so far have been carried out using $\mathcal{N}=0$ and $\mathcal{N}=1$ Liouville field theories.

In this talk we would like to present our recent studies of $\mathcal{N}=2$ Liouville theory based on the modular bootstrap approach [5, 6]. We recall that $\mathcal{N}=2$ Liouville theory consists of a complex boson $\phi+iY$ and conjugate fermi fields ψ^+,ψ^-. Here ϕ denotes the standard Liouville field coupled to the background charge $\mathcal{Q}$ and Y is a compact boson which is not coupled to background charge. In our convention $\mathcal{N}=2$ Liouville system has a central charge

$$c = 3 + 3\mathcal{Q}^2 \tag{1}$$

and the conformal dimension of a vertex operator $e^{\alpha\phi}$ is given by $\frac{1}{2}\alpha(\mathcal{Q}-\alpha)$.

Interests in $\mathcal{N}=2$ Liouville theory come from various directions:

1. Applications to string theory compactified on singular Calabi-Yau manifolds, NS5 branes etc.. It is well-known that in the CHS background of the NS5 brane, transverse directions to the 5-brane are described by the Liouville field $\times$ $SU(2)$ WZW model[7, 8, 9]. We then have
$$R^{5,1}\times R_\phi\times SU(2)_{\mathrm{WZW}} \approx R^{5,1}\times \underbrace{R_\phi\times U(1)}_{\mathcal{N}=2\ \text{Liouville}} \times \underbrace{SU(2)/U(1)}_{\mathcal{N}=2\ \text{minimal}} \tag{2}$$
where the first $U(1)$ factor is identified as the Y field. Thus the space-time of NS5 brane is described by the $\mathcal{N}=2$ Liouville field coupled to $\mathcal{N}=2$ minimal models. It is known that this geometry is T-dual to the ALE spaces and we will later compute the elliptic genus of ALE spaces by making use of the above description.

2. It is known that $\mathcal{N}=2$ Liouville theory is T-dual to $SL(2;R)/U(1)$ super- coset theory [10, 11] which describes the cigar geometry of 2-dimensional black hole
$$SL(2;R)/U(1)\ \text{theory} = \text{T-dual of}\ \mathcal{N}=2\ \text{Liouville}. \tag{3}$$
Thus we can discuss 2d black holes by using $\mathcal{N}=2$ Liouville theory. We will see below that unitary representations of $SL(2;R)/U(1)$ Kazama-Suzuki model are in fact identical to those of $\mathcal{N}=2$ Liouville theory [12, 6] and this gives a most straightforward proof of T-duality between these theories.

3. Bosonic sector of $\mathcal{N}=2$ Liouville theory consists of fields ϕ and Y and thus is identical to that of Liouville theory coupled to $c=1$ matter. Thus one expects a close relationship of $\mathcal{N}=2$ Liouville to bosonic models like sine-Liouville fields [13].

It is known that there are two different approaches to quantum Liouville theory;

1. conformal bootstrap,
2. modular bootstrap.

Conformal bootstrap [14, 15, 16] is a detailed and complex analysis of the Liouville system based on conformal invariance and bootstrap in the presence of world-sheet boundaries. It gives a detailed information on the dynamics of Liouville field and in particular led to the discovery of boundary states and D-branes in Liouville theory. On the other hand, the modular bootstrap which is based on the representation theory and modular properties of character formulas usually plays a secondary role in checking the consistency of the results of conformal bootstrap. In $\mathcal{N}=2$ Liouville theory, however, it is difficult to carry out the conformal bootstrap approach due to technical complexity and we instead propose to use modular bootstrap in order to bypass the technical difficulties.

Below we first discuss the representations of $\mathcal{N}=2$ Liouville theory and then construct boundary states and their wave functions using of the modular properties of the character formulas. It turns out that there exist three different types of boundary states in $\mathcal{N}=2$ Liouville: class 1 branes which are the analogues of ZZ branes in $\mathcal{N}=0$ theory [14], class 2 branes which are the analogues of FZZT branes [15, 16] and additional class 3 branes which come from the $U(1)$ degrees of freedom of $\mathcal{N}=2$ theory.

We then use the relationship of $\mathcal{N}=2$ Liouville and $SL(2;R)/U(1)$ theory and compare class 1,2,3 branes with the known D0,D1,D2 branes in the 2d black hole geometry [17]. We find good agreements between the two except for a subtle discrepancy in the case of D1 brane.

In $\mathcal{N}=0$ and 1 Liouville theories there exists the so-called duality symmetry of the theory where we exchange the parameter b with $1/b$ where b is related to the background charge $\mathcal{Q}$ as $\mathcal{Q}=b+1/b$. In the case of $\mathcal{N}=2$ theory an obvious duality symmetry is missing. Instead there exist two different types of Liouville potential terms

$$\text{F term}: \qquad \int d^2\theta e^{\frac{1}{\mathcal{Q}}\Phi}+\int d^2\bar{\theta} e^{\frac{1}{\mathcal{Q}}\bar{\Phi}}, \tag{4}$$

$$\text{D term}: \qquad \int d^4\theta e^{\frac{\mathcal{Q}}{2}(\Phi+\bar{\Phi})} \tag{5}$$

where Φ denotes a chiral superfield with its lowest component $\phi+iY$. These are both marginal operators preserving $\mathcal{N}=2$ superconformal symmetry and appear to play a dual role in the theory [18]. We will consider the issue of duality in $\mathcal{N}=2$ theory and point out that at a particular value of $\mathcal{Q}=1$ describing the singular K3 surface of A_1 type, this duality corresponds to the hyperKähler rotation of the K3 surface.

2 $\mathcal{N}=0$ theory

Now let us first briefly recall the results of $\mathcal{N}=0$ Liouville theory and introduce the idea of modular bootstrap. In $\mathcal{N}=0$ theory the central charge and the conformal dimension of a vertex operator is given by (we use the convention $\alpha'=1$)

$$c=1+6\mathcal{Q}^2, \quad h[e^{2\alpha\phi}]=\alpha(\mathcal{Q}-\alpha). \tag{6}$$

When $\alpha = \mathcal{Q}/2 + ip$, the vertex operator $e^{2\alpha\phi}$ describes a primary field of continuous representation with momentum p with dimension $h = p^2 + \mathcal{Q}^2/4$.

In $\mathcal{N} = 0$ theory there are basically two different representations;

1. Identity representation

$$\chi_{h=0}(\tau) = \frac{1-q}{\eta(\tau)} \cdot q^{-\frac{1}{4}(b+\frac{1}{b})^2}, \quad \mathcal{Q} = b + \frac{1}{b}. \tag{7}$$

2. Continuous representations

$$\chi_p(\tau) = \frac{q^{p^2}}{\eta(\tau)}. \tag{8}$$

Identity representation has a singular vector at level 1 while continuous representation is a non-degenerate representation. S-transform of these characters are given by

$$\chi_{h=0}(\frac{-1}{\tau}) = 4\sqrt{2}\int_0^\infty dp \sinh(2\pi bp)\sinh(\frac{2\pi p}{b})\chi_p(\tau), \tag{9}$$

$$\chi_p(\frac{-1}{\tau}) = 2\sqrt{2}\int_0^\infty dp \cos(2\pi pp')\chi_p(\tau). \tag{10}$$

Now we introduce the boundary states $|ZZ\rangle$ and $|FZZT;p\rangle$ and identify the characters as expectation values of the evolution operator

$$\chi_{h=0}(\frac{-1}{\tau}) = \langle ZZ|e^{i\pi\tau H^{(c)}}|ZZ\rangle, \tag{11}$$

$$\chi_p(\frac{-1}{\tau}) = \langle FZZT;p|e^{i\pi\tau H^{(c)}}|ZZ\rangle. \tag{12}$$

Here $|ZZ\rangle$ and $|FZZT;p\rangle$ denote ZZ and FZZT branes respectively and $H^{(c)}$ is the closed string Hamiltonian. These boundary states are expanded in terms of Ishibashi states as

$$|ZZ\rangle = \int_0^\infty dp' \Psi_0(p')|p'\rangle\rangle, \tag{13}$$

$$|FZZT;p\rangle = \int_0^\infty dp' \Psi_p(p')|p'\rangle\rangle \tag{14}$$

where $|p'\rangle\rangle$'s are Ishibashi states with momentum p' and $h = {p'}^2 + \mathcal{Q}^2/4$, and

$$\langle\langle p|e^{i\pi\tau H^{(c)}}|p'\rangle\rangle = \delta(p-p')\chi_p(\tau). \tag{15}$$

Here $\Psi_0(p')$ and $\Psi_p(p')$ are the disk one-point functions of ZZ and FZZT branes with a vertex operator $\exp{(\mathcal{Q}/2 + ip')\phi}$ insertion.

In terms of these wave functions (11),(12) are rewritten as

$$\chi_{h=0}(\frac{-1}{\tau}) = \int_0^\infty dp|\Psi_0(p)|^2\chi_p(\tau), \tag{16}$$

$$\chi_p(\frac{-1}{\tau}) = \int_0^\infty dp'\, \Psi_0(p')\Psi_p^*(p')\chi'_p(\tau). \tag{17}$$

By comparing with (9),(10) one finds

$$\Psi_0(p)\Psi(p)_0^* = 4\sqrt{2}\sinh(2\pi bp)\sinh(\frac{2\pi p}{b}), \tag{18}$$

$$\Psi_p(p')^*\Psi_0(p') = 2\sqrt{2}\cos(2\pi pp'). \tag{19}$$

Thus we can determine the wave functions

$$\Psi_0(p) = -2^{5/4}\cdot\frac{2\pi ip\hat{\mu}^{\frac{ip}{b}}}{\Gamma(1+i2pb)\Gamma(1+\frac{i2p}{b})}, \tag{20}$$

$$\Psi_p(p') = 2^{1/4}\frac{\hat{\mu}^{\frac{ip'}{b}}}{2\pi ip'}\Gamma(1-2ibp')\Gamma(1-\frac{2ip'}{b})\cos(2\pi pp'). \tag{21}$$

Here we have inserted some phase factors ($\hat{\mu}$ is related to the cosmological constant μ as $\hat{\mu} = \pi\mu\gamma(b^2)$ where $\gamma(x) = \Gamma(x)\Gamma(1-x)$).

This is the derivation of disk amplitudes using the modular bootstrap method: it is much simpler when compared with the computations in conformal bootstrap. Actually by the modular bootstrap method phase factors of the wave functions can not be determined, however, they cancel in the computation of cylinder amplitudes. Thus using modular bootstrap method we can analyze the Cardy condition and determine consistent boundary states.

Lessons we learn from the above discussions are as follows:

1. Identity representation does not appear in the closed string channel and closed strings states are spanned by the continuous representations $|p\rangle\rangle$. Conformal dimensions of continuous representations are bounded from below

$$h(p) = p^2 + \frac{\mathcal{Q}^2}{4} \geq \frac{\mathcal{Q}^2}{4} \tag{22}$$

and thus the closed string sector has a gap in the spectrum. This corresponds to the decoupling of gravity in the linear ditaton background.

2. On the other hand, the identity representation does occur in the open string channel.

3. One can check the consistency of the results of modular bootstrap with the conformal bootstrap analysis. For instance, one can check the reflection property of the disk amplitude

$$\Psi_0(-p) = R(p)\Psi_0(p) \tag{23}$$

where $R(p)$ denotes the reflection amplitude

$$R(p) = -\hat{\mu}^{-2ip/b}\frac{\Gamma(1+2ip/b)\Gamma(1+2ipb)}{\Gamma(1-2ip/b)\Gamma(1-2ipb)}. \tag{24}$$

3 $\mathcal{N}=2$ theory

Now we turn to the $\mathcal{N}=2$ theory. In the following we use the parametrization

$$\hat{c}=\frac{c}{3}=1+\mathcal{Q}^2, \quad \mathcal{Q}^2=\frac{2K}{N}, \quad K,N\in\mathbf{Z}_{\geq 1}, \tag{25}$$

and the conformal dimension of a vertex operator is give by

$$h[e^{\alpha\Phi}]=\frac{p^2}{2}+\frac{\mathcal{Q}^2}{8} \text{ for } \alpha=\frac{\mathcal{Q}}{2}+ip. \tag{26}$$

(we use the convention $\alpha'=2$). It is known that in $\mathcal{N}=2$ superconformal theory there exist three types of unitary representations, i.e. identity, continuous and discrete representations. We denote their character formulas as $ch_{id}(\tau;z)$,$ch_{cont}(\tau;z)$ and $ch_{dis}(\tau;z)$, respectively. Angular variable z is coupled to the $U(1)$ charge.

A continuous representation is a non-degenerate representation while the identity and discrete representation possesses a fermionic singular vector. Identity representation has also a bosonic singular vector. In the context of siring compactification we call them as graviton (identity), massive (continuous) and massless matter (discrete) representations.

For the application to string compactification it is necessary to project the theory onto states with integral $U(1)_R$ charges. In order to ensure charge integrality condition we take the sum over spectral flows of irreducible characters and introduce the extended characters

$$\chi_*(\tau;z)=\sum_{n\in r+N\mathbf{Z}} q^{\frac{\hat{c}}{2}n^2}e^{2i\pi\hat{c}zn}ch_*(\tau;z+n\tau), \;\; r\in\mathbf{Z}_N, \;\; *=id,\, cont,\, dis. \tag{27}$$

Note that we actually consider mod N spectral flow to assure good modular properties of extended characters. Parameter r runs over the range $0,1,\cdots,N-1$.

Extended characters are parametrized as

1. Identity representations :

$$\chi_{id}(m;\tau); \qquad m=2Kr, \;\; r\in\mathbf{Z}_N, \tag{28}$$

$$h=\frac{K}{N}r^2+|r|-\frac{1}{2}, \; Q=\frac{m}{N}+\mathrm{sign}(r)\cdot 1, \;\; r\neq 0 \tag{29}$$

$$h=Q=0, \;\; r=0.$$

2. Continuous representations :

$$\chi_{cont}(p,m;\tau); \qquad p\geq 0, \;\; m=2Kr,\, r\in\mathbf{Z}_N, \tag{30}$$

$$h=\frac{p^2}{2}+\frac{m^2+K^2}{4NK}, \; Q=\frac{m}{N}. \tag{31}$$

3. Discrete representations :

$$\chi_{dis}(s,r;\tau); \qquad m=s+2Kr, \;\; r\in\mathbf{Z}_N, \; 0\leq s\leq N+2K-1, \tag{32}$$

$$h=\frac{Kr^2+(r+\frac{1}{2})s}{N}, \qquad Q=\frac{m}{N}, \qquad 0\leq r\leq\frac{N}{2}, \tag{33}$$

$$h=\frac{Kr^2+(r+\frac{1}{2})s}{N}-(r+\frac{1}{2}), \; Q=\frac{m}{N}-1, \; \frac{-N}{2}\leq r\leq -1.$$

h and Q give the dimension and $U(1)$ charge of the representations. Note that the quantum number m is related to the $U(1)$ charge Q as $Q = m/N$ (mod integer). Explicit form of these characters are given in [5]. In the following we consider the NS sector of the theory. Characters in other sectors are obtained by half spectral flows. S-transformations of NS characters are given by

Identity representations:

$$\chi_{id}(m; -\frac{1}{\tau}) =$$
$$\frac{1}{\sqrt{2NK}} \sum_{m' \in Z_{2NK}} e^{-2\pi i \frac{mm'}{2KN}} \int_0^\infty dp' \frac{\sinh(\pi \mathcal{Q} p') \sinh(2\pi \frac{p'}{\mathcal{Q}})}{|\cosh \pi(\frac{p'}{\mathcal{Q}} + i\frac{m'}{2K})|^2} \chi_{cont}(p', m'; \tau)$$
$$+\frac{2}{N} \sum_{r' \in Z_N} \sum_{s'=K+1}^{N+K-1} \sin(\frac{\pi(s'-K)}{N}) e^{-2\pi i \frac{m(s'+2Kr')}{2KN}} \chi_{dis}(s', r'; \tau) \tag{34}$$

Continuous representations:

$$\chi_{cont}(p, m; -\frac{1}{\tau}) = \sqrt{\frac{2}{NK}} \sum_{m' \in Z_{2NK}} e^{-2\pi i \frac{mm'}{2NK}} \int_0^\infty dp' \cos(2\pi pp') \chi_{cont}(p', m'; \tau) \tag{35}$$

Discrete representations:

$$\chi_{dis}(s, r; -\frac{1}{\tau})$$
$$= \frac{1}{\sqrt{2NK}} \sum_{m' \in \mathbf{Z}_{2NK}} e^{-2\pi i \frac{(s+2Kr)m'}{2NK}} e^{\frac{i\pi m'}{2K}} \int_{-\infty}^\infty dp' \frac{e^{-2\pi(\frac{s-K}{N} - \frac{1}{2})\frac{p'}{\mathcal{Q}}}}{2\cosh \pi(\frac{p'}{\mathcal{Q}} + \frac{im'}{2K})} \chi_{cont}(p', m'; \tau)$$
$$+\frac{i}{N} \sum_{r' \in \mathbf{Z}_N} \sum_{s'=K+1}^{N+K-1} e^{-2\pi i \frac{(s+2Kr)(s'+2Kr') - (s-K)(s'-K)}{2NK}} \chi_{dis}(s', r'; \tau)$$
$$+\frac{i}{2N} \sum_{r' \in \mathbf{Z}_N} e^{-2\pi i \frac{(s+2Kr)(s'+2Kr')}{2KN}} \{\chi_{dis}(s' = K, r'; \tau) - \chi_{dis}(s' = N+K, r'; \tau)\} \tag{36}$$

Note that contributions from the "boundaries" of the range of s, $K \leq s \leq K+N$ appear in the RHS of (37).

The above transformation law has a peculiar structure,

$$\text{(continuous rep)} \overset{S}{\Longrightarrow} \text{(continuous rep)} \tag{37}$$
$$\text{(idenitity or discrete rep)} \overset{S}{\Longrightarrow} \text{(continuous rep)} + \text{(discrete rep)} \tag{38}$$

Only a part of discrete representations appear in the RHS (38). Such a pattern of modular transformations was first observed in the representation theory of $\mathcal{N} = 4$ of superconformal algebra [19].

1. We note that there appear no identity representations in the RHS of above formulas (34), (35). (37).

2. Only the discrete representations in the range $K \leq s \leq N+K$ appear in the RHS.

3. It is still possible to show that S-transformation squared equals C, $S^2 = C$ where C is the charge conjugation operation [20]. This happens because the shift of the momentum contour becomes necessary in the 2nd S-transform and from the fact that identity and discrete representations can be obtained from continuous representations at some complex values of the momentum.

Based on the above transformation law we propose that in the $\mathcal{N}=2$ theory closed string Ishibashi states are spanned by

$$\text{(continuous reps)+(discrete reps with } K \leq s \leq K+N). \tag{39}$$

As we shall see below this spectrum agrees with those of the $SL(2;R)/U(1)$ coset theory.

We can construct class 1, 2, 3 boundary states corresponding to identity, continuous and discrete representations using the modular transformation law. In the case of class 1 and 2 branes, boundary wave functions can be easily read off from the modular S matrices. Class 1 boundary state (for the case $m=0$) is constructed as

$$|B;id\rangle = \int_0^\infty dp' \sum_{m' \in \mathbf{Z}_{2NK}} \Psi_{id}(p',m')|p',m'\rangle\rangle + \sum_{r' \in \mathbf{Z}_N} \sum_{s'=K+1}^{N+K-1} C_{id}(s',r')|s',r'\rangle\rangle \tag{40}$$

where

$$\Psi_{id}(p',m') = \frac{1}{\mathcal{Q}} \cdot \left(\frac{2}{NK}\right)^{1/4} \cdot \frac{\Gamma(\frac{1}{2}+\frac{m'}{2K}+i\frac{p'}{\mathcal{Q}})\Gamma(\frac{1}{2}-\frac{m'}{2K}+i\frac{p'}{\mathcal{Q}})}{\Gamma(i\mathcal{Q}p')\Gamma(1+i\frac{2p'}{\mathcal{Q}})}, \tag{41}$$

$$C_{id}(s',r') = \left(\frac{2}{N}\right)^{1/2} \sqrt{\sin\frac{\pi(s'-K)}{N}} \tag{42}$$

and $|p,m\rangle\rangle$ and $|r,s\rangle\rangle$ are Ishibashi states of continuous and discrete representations

$$\langle\langle p,m|e^{i\pi\tau H^{(c)}}|p',m'\rangle\rangle = \delta(p-p')\delta_{m,m'}\chi_{cont}(p,m;\tau), \tag{43}$$

$$\langle\langle s,r|e^{i\pi\tau H^{(c)}}|s',r'\rangle\rangle = \delta_{r,r'}\delta_{s,s'}\chi_{dis}(s,r;\tau). \tag{44}$$

We find that the amplitude $\langle B;id|\exp(i\pi\tau H^{(c)})|B;id\rangle$ reproduces the RHS of (34). Class 2 states are constructed in a similar manner. By computing various cylinder amplitudes one can check the Cardy consistency conditions.

On the other hand, in the case of class 3 branes the situation becomes somewhat delicate due to the presence of boundary terms in modular transformation (37). In order to cancel the effect of these terms one has to consider a pair of discrete representations $[(r_1,s_1),(r_2,s_2)]$ with $(s_1+2r_1K)-(s_2+2r_2K) = \mathbf{N} \times$ odd integer, and a combination of characters $\chi_{dis}(r_1,s_1;\tau) + \chi_{dis}(r_2,s_2;\tau)$. In this case Cardy condition are not necessarily obeyed depending on the parameters K,N of the theory.

4 $SL(2;R)/U(1)$ theory

It is well-known that $SL(2;R)_k/U(1)$ supercoset theory is T-dual to the $\mathcal{N}=2$ Liouville. Central charges of these theories read as

$$\hat{c} = 1 + \frac{2}{k} = 1 + \mathcal{Q}^2 = 1 + \frac{2K}{N}. \tag{45}$$

Thus the level k is related to the parameters K, N by

$$k = \frac{N}{K}. \tag{46}$$

By studying the representations of $SL(2;R)/U(1)$ Kazama-Suzuki model, one finds that they are in fact in an one-to-one correspondence with those of $\mathcal{N}=2$ Liouville and their character formulas are identical [6]. (Relation of character formulas of bosonic $SL(2;R)/U(1)$ theory to those of $\mathcal{N}=2$ Liouville is given in [20]). Correspondence of parameters is given by

$$\text{identity rep}: \qquad j = 0, \tag{47}$$

$$\text{continuous rep}: \qquad j = \frac{1}{2} + i\frac{p}{\mathcal{Q}}, \tag{48}$$

$$\text{discrete rep}: \qquad j = \frac{s}{2K} \tag{49}$$

where j denotes the spin of $SL(2;R)$ representation.

Path integral evaluation of toroidal partition function of $SL(2;R)/U(1)$ theory [21] shows that the theory contains continuous representations as well as discrete representations in the (improved) unitary range [22, 6, 23]

$$\frac{1}{2} \leq j \leq \frac{k+1}{2}. \tag{50}$$

This agrees with the range $K \leq s \leq K+N$ of $\mathcal{N}=2$ Liouville under the correspondence $j = s/2K$.

$SL(2;R)/U(1)$ theory describes the cigar geometry of 2d black hole and its boundary states are constructed by using Born-Infeld action and the results of $SL(2;R)$ theory [17]. One can compare the results of $\mathcal{N}=2$ theory with those of $SL(2;R)/U(1)$ [24, 25].

Comparison of Boundary sates:

Type of branes	$\mathcal{N}=2$ Liouville		Type of branes	$SL(2;R)/U(1)$
A	class 1 identity rep		B	D0 brane localized at tip of cigar
B	class 2' continuous rep		A	D1 brane extended along radial direction
A	class 3 discrete rep		B	D2 brane wraps the whole cigar
A	class 2 continuous rep		B	D2 brane wraps a part of cigar

There exists an overall good agreement between the two. Wave functions of class 1 branes agree with those of $SL(2;R)/U(1)$ theory and also with the results of conformal bootstrap [26, 27]. We also have a match of class 3 and D2 branes with the identification

$$\sigma = \pi(\frac{s-1}{k} - \frac{1}{2}) \tag{51}$$

where σ is related to the gauge field strength on D2 brane [17]. Consistency of overlap of two D2 branes with σ, σ' requires

$$\sigma - \sigma' = 2\pi n \frac{1}{k} \tag{52}$$

where n is an integer related to the D0 brane charges. This condition immediately follows from the above identification (51).

However, in the case of class 2 branes there is a delicate discrepancy with the D1 brane of $SL(2;R)/U(1)$ theory. Their wave functions have a form

$$\Psi^{class2}_{p,m}(p', m') \approx f(p', m') \cos(2\pi p p') \tag{53}$$

$$\Psi^{D1}_{p,m}(p', m') \approx f(p', m') \frac{e^{2\pi i p p'} + (-1)^{m'} e^{-2\pi i p p'}}{2}. \tag{54}$$

Thus D1 brane has an extra phase factor $(-1)^{m'}$ as compared with the class 2 brane. Authors of [25] call (54) as the class 2' brane. The extra phase factor is related to different reflection amplitudes for A and B type states. Type B class 2 brane in fact does not have a good semi-classical limit unlike class 2' branes. It is suggested that type A class 2 branes instead are well-defined and correspond to partially wrapped D2 branes in $SL(2;R)/U(1)$ theory [25].

5 Singular Calabi-Yau manifolds

When $\mathcal{N} = 2$ Liouville theory is coupled to $\mathcal{N} = 2$ minimal models,

$$|\text{class 1 boundary state of Liouville}\rangle \otimes |\text{boundary states of minimal model}\rangle \tag{55}$$

describe vanishing cycles of singular Calabi-Yau manifolds: one recovers the correct intersection numbers by computing open string Witten index. See [5] for details.

One can also compute the elliptic genus of singular Calabi-Yau manifolds using $\mathcal{N} = 2$ Liouville theory [6]. Elliptic genus is defined by

$$Z(\tau; z) = Tr(-1)^{F_R} e^{2\pi i J_0 z} q^{L_0 - \hat{c}/8} \bar{q}^{\bar{L}_0 - \hat{c}/8} \tag{56}$$

and is invariant under smooth deformation of the parameters of the theory. Here F_R denotes the fermion number of the right-moving sector. In the case of compact manifolds elliptic genus is known to have a good modular property and is a (quasi) Jacobi-form. It turns out that this is no longer the case in non-compact manifolds.

For instance, in the case of conifold one obtains the elliptic genus

$$Z_{conifold}(\tau, z) = \frac{1}{2} \frac{\theta_1(\tau; 2z)}{\theta_1(\tau; z)}. \tag{57}$$

In the case of ALE space with A_1 singularity it is given by

$$Z_{ALE(A_1)}(\tau,z) = -ch_0^{\mathcal{N}=4,\tilde{R}}(\ell=0;\tau;z). \tag{58}$$

Here $ch_0^{\mathcal{N}=4}(\ell=0)$ denotes the massless $\mathcal{N}=4$ character of isospin=0. Elliptic genera of general A_{n-1} type singularity are described in terms of the Appell function [6]

$$K_\ell(\tau,\nu,\mu) = \sum_{m\in Z} \frac{e^{i\pi m^2 \ell\tau + 2\pi i m\ell\nu}}{1-e^{2\pi i(\nu+\mu+m\tau)}}. \tag{59}$$

Appell function is closely related to the character of discrete representations of $\mathcal{N}=2$ theory [28] and describe holomorphic sections of higher rank vector bundles on Riemann surfaces [29].

6 $\mathcal{Q}=1$ and Duality

When the background charge takes a special value $\mathcal{Q}=1$, dimension of the target space $\hat{c} = 1+\mathcal{Q}^2$ becomes 2 and $\mathcal{N}=2$ Liouville theory describes string compactification on ALE space with A_1 singularity. Thus at $\mathcal{Q}=1$ $\mathcal{N}=2$ SUSY is enhanced to $\mathcal{N}=4$. Generators of $\mathcal{N}=2$ superconformal algebra (SCA) at $\mathcal{Q}=1$ are given by

$$T = -\frac{1}{2}\left[(\partial Y)^2 + (\partial\phi)^2 + \partial^2\phi + (\psi^+\partial\psi^- - \partial\psi^+\psi^-)\right], \tag{60}$$

$$G^+ = \frac{-1}{\sqrt{2}}\psi^+(i\partial Y + \partial\phi) - \frac{1}{\sqrt{2}}\partial\psi^+, G^- = \frac{-1}{\sqrt{2}}\psi^-(i\partial Y - \partial\phi) + \frac{1}{\sqrt{2}}\partial\psi^-, \tag{61}$$

$$J_{U(1)} = \psi^+\psi^- - i\partial Y = i\partial H - i\partial Y \tag{62}$$

where we have bosonized the fermi fields $\psi^+\psi^- = i\partial H$. $SU(2)$ currents of $\mathcal{N}=4$ SCA are given by

$$J^+_{SU(2)} = e^{iH-iY}, \quad J^-_{SU(2)} = e^{-iH+iY}, \quad J^3_{SU(2)} = \frac{i}{2}(\partial H - \partial Y). \tag{63}$$

At $\mathcal{Q}=1$ Liouville potential terms (4),(5) become

$$S_+ = \int d^2z\, e^{-\phi-iY-iH},\ S_3 = \int d^2z\,(-i\partial Y - i\partial H)\,(i\bar\partial Y + i\bar\partial H)e^{-\phi},\ S_- = \int d^2z\, e^{-\phi+iY+iH}. \tag{64}$$

We find that $S_\pm, S_3$ form a triplet under a new $SU(2)$ algebra $SU(2)'$ generated by

$$J^+_{SU(2)'} = e^{iH+iY},\ J^-_{SU(2)'} = e^{-iH-iY},\ J^3_{SU(2)'} = \frac{1}{2}(i\partial H + i\partial Y). \tag{65}$$

$SU(2)'$ commutes with $SU(2)$ of $\mathcal{N}=4$ SCA and we have an $SU(2)\times SU(2)'$ structure

$$\begin{array}{ccc} G^+ = G^{+,+} & \xrightarrow{J^-_{SU(2)'}} & G^{+,-} \\ {\scriptstyle J^-_{SU(2)}}\downarrow & & \uparrow{\scriptstyle J^+_{SU(2)}} \\ G^{-,+} & \xleftarrow{J^+_{SU(2)'}} & G^{-,-} = G^- \end{array} \tag{66}$$

$$G^{+,-} = +\frac{1}{\sqrt{2}}e^{-iY}(\partial\phi - \partial H) + \frac{1}{\sqrt{2}}\partial e^{-iY}, \tag{67}$$

$$G^{-,+} = -\frac{1}{\sqrt{2}}e^{-iY}(\partial\phi + \partial H) - \frac{1}{\sqrt{2}}\partial e^{-iY}. \tag{68}$$

Action of $SU(2)'$ transforms $G^{\pm}$ into $G^{\mp}$ and induces the change of complex structure. Thus it corresponds to the hyperKähler rotations of K3 surface. Since $S_{\pm}$ an S_3 are transformed into each other under $SU(2)'$, the $\mathcal{N}=2$ duality amounts to a hyperKähler rotation at $\mathcal{Q}=1$. More details will be discussed in [30].

After this paper has been presented at strings2004 a new preprint appeared [31] where new results of conformal bootstrap of $\mathcal{N}=2$ theory are reported.

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